

CONVERGENCE OF PARTIAL MAPS VIA BORNOLGY THROUGH IDEAL AND ITS CHARACTERIZATION

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ABSTRACT. In this paper we consider the idea of I - convergence of nets of partial function from a metric space (X, d) to a metric space (Y, μ) and derive several basic characterization. This idea extends the concept of convergence of nets of partial function introduced by G. Beer et.al [1].

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1. Introduction

We mean by a partial map or a partial function from a metric space (X, d) to a metric space (Y, μ) , a pair (D, u) where D is a nonempty closed subset of X and $u : D \rightarrow Y$ is a function. Let us denote the set of all such partial maps by $\mathcal{P}[X, Y]$ and by $\mathcal{C}[X, Y]$ we mean those partial maps which are continuous on their respective domains. The notion of convergence of partial maps was first introduced by G. Beer et.al.[1]. The notion of I -convergence of nets was first introduced by B. K. Lahiri and P. Das [8]. In this paper we will use this two notion and introduced a new type of convergence on partial maps which will produced a new research area.

2. Preliminaries

In this section we give some basic definitions and discuss some ideas which will helpful to understand this paper in the next section.

Definition 2.1. [] If X is a non-void set, then a family $I \subset 2^X$ is called an ideal if

- (i) $\phi \in I$ and

- (ii) $A ; B \in I$ implies $A \cup B \in I$ and
- (iii) $A \in I ; B \subset A$ implies $B \in I$.

The ideal I is called non-trivial if $I \neq \{\phi\}$ and $X \notin I$.

Definition 2.2. [] A non empty family \mathbb{F} of subsets of a non-void set X is a filter if

- (i) $\phi \notin \mathbb{F}$ and
- (ii) $A ; B \in \mathbb{F}$ implies $A \cap B \in \mathbb{F}$ and
- (iii) $A \in \mathbb{F} ; A \subset B$ implies $B \in \mathbb{F}$.

Clearly $I \subset 2^X$ is a non-trivial ideal of X if and only if $\mathbb{F} = \mathbb{F}(I) = \{A \subset X : X \setminus A \in I\}$ is a filter on X , called the filter associated with I .

The following two definitions are well known to all but for sake of completeness we give it below

Definition 2.3. Let Γ be a non-void set and let \geq be a binary relation on Γ such that \geq is reflexive, transitive and for any two elements $m ; n \in \Gamma$, there is an element $p \in \Gamma$ such that $p \geq m$ and $p \geq n$. The pair $(\Gamma; \geq)$ is called a directed set.

Definition 2.4. Let $(\Gamma; \geq)$ be a directed set and let X be a non-void set. A mapping $\gamma : \Gamma \rightarrow X$ is called a net in X denoted by $\{\gamma_n : n \in \Gamma\}$ or simply by $\{\gamma_n\}$ when the set Γ is clear.

Throughout the paper $X = (X, d)$ and $Y = (Y, \mu)$ will denote metric spaces. We write $CL(X)$ for the collection of the closed nonempty subsets of X , $K(X)$ is the collection of the compact nonempty subsets of X . And by \mathbb{N} we denote the set of all natural numbers and I will denote a non-trivial ideal of a directed set Γ .

For $n \in \Gamma$ let $M_n = \{k \in \Gamma : k \geq n\}$. Then the collection

$$\mathbb{F}_0 = \{A \subset \Gamma : A \supset M_n \text{ for some } n\}$$

forms a filter in γ . Let $I_0 = \{A \subset \Gamma : \Gamma \setminus A \in \mathbb{F}_0\}$. Then I_0 is also a non-trivial ideal in Γ .

Definition 2.5. A non-trivial ideal I of Γ will be called $D - admissible$ if $M_n \in \mathbb{F}(I)$ for all $n \in \Gamma$.

We now discuss the notion of bornology (for more details see [])
If $x_0 \in X$ and $\epsilon > 0$, $B(x_0, \epsilon)$ is the open ϵ -ball with center x_0 and radius ϵ . If A is a nonempty subset of X , we write $d(x_0, A)$ for the distance from x_0 to A . We denote by A^ϵ the ϵ -enlargement of the set A :

$$A^\epsilon = \{x : d(x, A) < \epsilon\} = \bigcup_{x \in A} B(x, \epsilon).$$

Definition 2.6. A *bornology* \mathcal{B} on a metric space (X, d) is a family of subsets of X , covering X , closed under taking finite unions, and hereditary, i.e., closed under taking nonempty subsets.

The smallest bornology on X is the family of the finite subsets of X , \mathcal{F} , and the largest is the family of all non empty subsets of X , $\mathcal{P}_0(X)$. Other important bornologies are: the family B_d of the nonempty d -bounded subsets, the family B_{tb} of the nonempty d -totally bounded subsets and the family \mathcal{K} of nonempty subsets of X whose closure sets are compact.

We now give some basic definition related to bornological convergence as defined in ([1], [2]).

Definition 2.7. Let (X, d) be a metric space and \mathcal{B} be a bornology on (X, d) . A net $\langle D_\gamma \rangle_{\gamma \in \Gamma}$ in $\mathcal{P}_0(X)$ is called \mathcal{B}^- -convergent (lower bornological convergent) to $D \in \mathcal{P}_0(X)$ if for every $B \in \mathcal{B}$ and $\epsilon > 0$, the following inclusion holds eventually:

$$D \cap B \subset D_\gamma^\epsilon.$$

In this case we shall write $D \in \mathcal{B}^- - \lim D_\gamma$ when this holds. Similarly the net is called \mathcal{B}^+ -convergent (upper bornological convergent) to $D \in \mathcal{P}_0(x)$ if for every $B \in \mathcal{B}$ and $\epsilon > 0$,

$$D_\gamma \cap B \subset D^\epsilon.$$

In this case we shall write $D \in \mathcal{B}^+ - \lim D_\gamma$ when this occurs.

Naturally two-sided bornological convergence occurs when both upper and lower convergences occur, and we then write $D \in \mathcal{B} - \lim D_\gamma$.

Definition 2.8. Let (X, d) , (Y, μ) be metric spaces, and \mathcal{B} be a bornology on X . Let Γ be a directed set and let $\langle \langle D_\gamma, u_\gamma \rangle \rangle_{\gamma \in \Gamma}$ be a net in $\mathcal{P}[X, Y]$. We say that the net is $\mathcal{P}(\mathcal{B})$ -convergent to (D, u) , we write $(D, u) \in \mathcal{P}(\mathcal{B}) - \lim \langle \langle D_\gamma, u_\gamma \rangle \rangle$, if for every $B \in \mathcal{B}$ and $\epsilon > 0$, the following two conditions hold for all indices $\gamma \geq \gamma_0$

- (i) for each nonempty subset B_1 of B , $u(D \cap B_1) \subset [U_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon$
- (ii) for each nonempty subset B_1 of B , $u_\gamma(D_\gamma \cap B_1) \subset [U(D \cap B_1^\epsilon)]^\epsilon$.

The most tangible and visual description of $\mathcal{P}(\mathcal{B})$ -convergence is the following: for each $B \in \mathcal{B}$ and $\epsilon > 0$, eventually both $Gr(u_\gamma) \cap (B \times Y) \subset Gr(u)^\epsilon$ and $Gr(u) \cap (B \times Y) \subset Gr(u_\gamma)^\epsilon$. In this formulation, the enlargement is taken with respect to any metric compatible with the product uniformity. For definiteness, we choose the *box metric* defined by

$$(d \times \mu)((x_1, y_1), (x_2, y_2)) := \max \{d(x_1, x_2), \mu(y_1, y_2)\}.$$

Definition 2.9. Let (X, d) , (Y, μ) be metric spaces, and \mathcal{B} be a bornology on X . Let Γ be a directed set and $\langle \langle D_\gamma, u_\gamma \rangle \rangle_{\gamma \in \Gamma}$ be a net in $\mathcal{P}[X, Y]$. We say that

the net is $\mathcal{P}^-(\mathcal{B})$ -convergent to (D, u) , we write $(D, u) \in \mathcal{P}^-(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$, if for every $B \in \mathcal{B}$ and $\epsilon > 0$,

$$\text{for all } B_1(\subset B), u(D \cap B_1) \subset [U_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon$$

holds. Similarly we can say the net is $\mathcal{P}^+(\mathcal{B})$ -convergent to (D, u) , we write $(D, u) \in \mathcal{P}^+(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$, if for every $B \in \mathcal{B}$ and $\epsilon > 0$,

$$\text{for all } B_1(\subset B), u_\gamma(D_\gamma \cap B_1) \subset [U(D \cap B_1^\epsilon)]^\epsilon$$

holds.

3. Main Results

Let (X, d) and (Y, μ) be metric spaces. In this section we investigate the notion of convergence of partial maps by ideals of directed sets. So we first give some definition.

Definition 3.1. Let (X, d) , (Y, μ) be metric spaces, and \mathcal{B} be a bornology on X . Let Γ be a directed set and I be a nontrivial ideal of Γ and $\langle \langle D_\gamma, u_\gamma \rangle \rangle_{\gamma \in \Gamma}$ be a net in $\mathcal{P}[X, Y]$. We say that the net is $\mathcal{P}_I(\mathcal{B})$ -convergent to (D, u) , we write $(D, u) \in \mathcal{P}_I(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$, if for every $B \in \mathcal{B}$ and $\epsilon > 0$, the following two criteria

- (i) $\{\gamma : \forall B_1(\subset B), u(D \cap B_1) \subset [U_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon\} \in \mathbb{F}(I)$.
- (ii) $\{\gamma : \forall B_1(\subset B), u_\gamma(D_\gamma \cap B_1) \subset [U(D \cap B_1^\epsilon)]^\epsilon\} \in \mathbb{F}(I)$

hold.

Definition 3.2. Let (X, d) be a metric space and \mathcal{B} be a bornology on (X, d) . A net $\langle D_\gamma \rangle_{\gamma \in \Gamma}$ in $\mathcal{P}_0(X)$ is called \mathcal{B}_I^- -convergent (lower bornological I -convergent) to $D \in \mathcal{P}_0(X)$ if for every $B \in \mathcal{B}$ and $\epsilon > 0$,

$$\{\gamma : D \cap B \subset D_\gamma^\epsilon\} \in \mathbb{F}(I). \text{ In this case we shall write } D \in \mathcal{B}_I^- - \lim D_\gamma$$

when this holds. Similarly the net is called \mathcal{B}_I^+ -convergent (upper bornological I -convergent) to $D \in \mathcal{P}_0(x)$ if for every $B \in \mathcal{B}$ and $\epsilon > 0$,

$$\{\gamma : D_\gamma \cap B \subset D^\epsilon\} \in \mathbb{F}(I). \text{ In this case we shall write } D \in \mathcal{B}_I^+ - \lim D_\gamma$$

when this occurs.

Definition 3.3. Let (X, d) , (Y, μ) be metric spaces, and \mathcal{B} be a bornology on X . Let Γ be a directed set and I be a nontrivial ideal of Γ and $\langle \langle D_\gamma, u_\gamma \rangle \rangle_{\gamma \in \Gamma}$ be a net in $\mathcal{P}[X, Y]$. We say that the net is $\mathcal{P}_I^-(\mathcal{B})$ -convergent to (D, u) , we write $(D, u) \in \mathcal{P}_I^-(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$, if for every $B \in \mathcal{B}$ and $\epsilon > 0$,

$$\{\gamma : \forall B_1(\subset B), u(D \cap B_1) \subset [U_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon\} \in \mathbb{F}(I).$$

holds. Similarly we can say the net is $\mathcal{P}_I^+(\mathcal{B})$ -convergent to (D, u) , we write $(D, u) \in \mathcal{P}_I^+(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$, if for every $B \in \mathcal{B}$ and $\epsilon > 0$,

$$\{\gamma : \forall B_1(\subset B), u_\gamma(D_\gamma \cap B_1) \subset [U(D \cap B_1^\epsilon)]^\epsilon\} \in \mathbb{F}(I)$$

holds.

Proposition 3.1. *Let $\langle \langle D_\gamma, u_\gamma \rangle \rangle_{\gamma \in \Gamma}$ be a net in $\mathcal{P}[X, Y]$, \mathcal{B} be a bornology on the metric space (X, d) and I is an ideal of Γ .*

- (i) *If $(D, u) \in \mathcal{P}_I^-(\mathcal{B}) - \lim (D_\gamma, u_\gamma)$, then $\forall B \in \mathcal{B}$ and $\forall \epsilon > 0$, $\{\gamma : D \cap B \subset D_\gamma^\epsilon\} \in \mathbb{F}(I)$.*
- (ii) *If $(D, u) \in \mathcal{P}_I^+(\mathcal{B}) - \lim (D_\gamma, u_\gamma)$, then $\forall B \in \mathcal{B}$ and $\forall \epsilon > 0$, $\{\gamma : D_\gamma \cap B \subset D^\epsilon\} \in \mathbb{F}(I)$.*

Proof. We only prove statement (i), one can prove statement (ii) similarly. Let $B \in \mathcal{B}$ and $\epsilon > 0$ be given. By assumption we have

$$A = \{\gamma : \forall B_1 (\subset B), u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon\} \in \mathbb{F}(I).$$

Since I is nontrivial, we can choose $\gamma_1 \in A$. Let $x \in D \cap B$. Now with $B_1 = \{x\}$ we get

$$u(x) \in [u_{\gamma_1}(D_{\gamma_1} \cap \{x\}^\epsilon)]^\epsilon.$$

This means that for some $v \in D_{\gamma_1} \cap B_d(x, \epsilon)$ we have $\mu(u(x), u_{\gamma_1}(v)) < \epsilon$. More particularly, $x \in B_d(v, \epsilon) \subset D_{\gamma_1}^\epsilon$. Since $x \in D \cap B$ is arbitrary thus

$A \subset \{D \cap B \subset D_\gamma^\epsilon\}$. As $A \in \mathbb{F}(I)$ thus the later set. This completes the proof. □

Proposition 3.2. *The condition for every $B \in \mathcal{B}$ and $\epsilon > 0$,*

$$\{\gamma : \forall B_1 (\subset B), u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\epsilon)]^\epsilon\} \in \mathbb{F}(I)$$

holds if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$,

$$\left\{ \gamma : \sup_{z \in D_\gamma \cap B} \inf_{x \in B_d(z, \epsilon)} \mu(u_\gamma(z), u_\gamma(x)) < \epsilon \right\} \in \mathbb{F}(I)$$

holds.

Proof. First we prove necessary part of the proposition. Let $B \in \mathcal{B}$ and $\epsilon > 0$ be given. Then by assumption we have

$$A = \left\{ \gamma : \forall B_1 (\subset B), u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^{\frac{\epsilon}{2}})]^{\frac{\epsilon}{2}} \right\} \in \mathbb{F}(I).$$

Let us choose $z \in D_\gamma \cap B$ with $B_1 = \{z\}$, $\gamma \in A$ we have $u_\gamma(z) \in [u(D \cap B_1^{\frac{\epsilon}{2}})]^{\frac{\epsilon}{2}}$. This means that for some $x \in D \cap B_d(z, \frac{\epsilon}{2})$ implies $\mu(u_\gamma(z), u(x)) < \frac{\epsilon}{2}$. Then clearly $\inf_{x \in D \cap B_d(z, \frac{\epsilon}{2})} \mu(u_\gamma(z), u(x)) < \frac{\epsilon}{2}$. Also

$$\sup_{z \in D_\gamma \cap B} \inf_{x \in D \cap B_d(z, \frac{\epsilon}{2})} \mu(u_\gamma(z), u(x)) \leq \frac{\epsilon}{2} < \epsilon.$$

But again

$$\sup_{z \in D_\gamma \cap B} \inf_{x \in D \cap B_d(z, \epsilon)} \mu(u_\gamma(z), u(x)) < \sup_{z \in D_\gamma \cap B} \inf_{x \in D \cap B_d(z, \frac{\epsilon}{2})} \mu(u_\gamma(z), u(x)) < \epsilon.$$

Thus $A \subset \left\{ \gamma : \sup_{z \in D_\gamma \cap B} \inf_{x \in D \cap B_d(z, \epsilon)} \mu(u_\gamma(z), u(x)) < \epsilon \right\}$. Since $A \in \mathbb{F}(I)$ thus the later one in $\mathbb{F}(I)$. Therefore the condition is necessary.

Now we prove the sufficient part of the proposition. Suppose

$$\{\gamma : \sup_{z \in D_\gamma \cap B} \inf_{x \in D \cap B_d(z, \frac{\epsilon}{2})} \mu(u_\gamma(z), u(x)) < \epsilon\} \in \mathbb{F}(I).$$

Clearly if $B_1 \subset B$, we have

$$A_1 = \{\gamma : \sup_{z \in D_\gamma \cap B_1} \inf_{x \in D \cap B_d(z, \frac{\epsilon}{2})} \mu(u_\gamma(z), u(x)) < \epsilon\} \in \mathbb{F}(I).$$

Let $\gamma \in A_1$ then we have for all $z \in D_\gamma \cap B_1$ there exists $x \in D \cap \{z\}^\epsilon \subset D \cap B_1^\epsilon$ with $\mu(u_\gamma(z), u(x)) < \epsilon$. Thus

$$A_1 \subset \{\gamma : \forall B_1(\subset B), u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\epsilon)]^\epsilon\}.$$

But $A_1 \in \mathbb{F}(I)$, thus the later set in $\mathbb{F}(I)$. Hence the condition is sufficient. \square

Similarly one can prove the following result.

Proposition 3.3. *The condition for every $B \in \mathcal{B}$ and $\epsilon > 0$,*

$$\{\gamma : \forall B_1(\subset B), u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon\} \in \mathbb{F}(I)$$

holds if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$,

$$\left\{ \gamma : \sup_{z \in D \cap B} \inf_{x \in B_d(z, \epsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) < \epsilon \right\} \in \mathbb{F}(I)$$

holds.

Theorem 3.4. *Let $\langle \langle D_\gamma, u_\gamma \rangle \rangle_{\gamma \in \Gamma}$ be a net of partial functions from the metric space (X, d) to the metric space (Y, μ) and let I be an ideal of Γ and \mathcal{B} be a bornology on X . Then for $(D, u) \in \mathcal{P}[X, Y]$*

- (1) $Gr(u) \in (B_I^*)^- - \lim Gr(u_\gamma)$ if and only if $(D, u) \in \mathcal{P}_I^-(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$;
- (2) $Gr(u) \in (B_I^*)^+ - \lim Gr(u_\gamma)$ if and only if $(D, u) \in \mathcal{P}_I^+(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$.

Proof. We just verify statement (1). Suppose $(D, u) \in \mathcal{P}_I^-(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$. To verify bornology convergence of graphs, it is suffices to work with the basic sets in \mathcal{B}^* . Let $B \times Y$ be such a basic set where $B \in \mathcal{B}$. Let $\epsilon > 0$ be given, we have by assumption

$$A = \{\gamma : D \cap B \subset D_\gamma^\epsilon\} \in \mathbb{F}(I) \text{ and}$$

$$B = \left\{ \gamma : \sup_{z \in D \cap B} \inf_{x \in B_d(z, \epsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) < \epsilon \right\} \in \mathbb{F}(I).$$

But $A, B \in \mathbb{F}(I) \Rightarrow A \cap B \in \mathbb{F}(I)$. Choose $\gamma \in A \cap B$ then both

$$(i) \ D \cap B \subset D_\gamma^\epsilon$$

$$(ii) \ \sup_{z \in D \cap B} \inf_{x \in B_d(z, \epsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) < \epsilon$$

holds. Also for $\gamma \in A \cap B$ and $(z, u(z)) \in (B \times Y) \cap Gr(u)$ so that $z \in D$. By (i) $B_d(z, \epsilon) \cap D_\gamma \neq \emptyset$ and by (ii) for some $x \in B_d(z, \epsilon) \cap D_\gamma$, we have $\mu(u_\gamma, u(z)) < \epsilon$. So we have $(x, u_\gamma(x)) \in Gr(u_\gamma)$ and $(d \times \mu)((z, u(z))(x, u_\gamma(x))) < \epsilon$ and this yields $Gr(u) \cap (B \times Y) \subset Gr^\epsilon(u_\gamma)$. Therefore

$$A \cap B \subset \{\gamma : Gr(u) \cap (B \times Y) \subset Gr^\epsilon(u_\gamma)\}.$$

Since $A \cap B \in \mathbb{F}(I)$, so the later set.

Conversely, we consider the lower bornological I -convergence of graphs. Let $B \in \mathcal{B}$ and $\epsilon > 0$ be given. Choosing $0 < \eta < \epsilon$, we have

$$A_1 = \{\gamma : Gr(u) \cap (B \times Y) \subset Gr^\eta(u_\gamma)\} \in \mathbb{F}(I).$$

Let $\gamma \in A_1$ and let $z \in D \cap B$ be arbitrary. Clearly $(z, u(z)) \in (B \times Y) \cap Gr(u)$, so there exists $(y_0, u_\gamma(y_0)) \in Gr(u_\gamma)$ such that

$$(d \times \mu)((z, u(z))(y_0, u_\gamma(y_0))) < \eta.$$

Thus $d(z, y_0) < \eta < \epsilon$. By the same argument we can say $\mu(u(z), u_\gamma(y_0)) < \eta$, so that $\inf_{x \in B_d(z, \epsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) < \eta$ and by taking the supremum over $z \in D \cap B$ we have

$$\sup_{z \in D \cap B} \inf_{x \in B_d(z, \epsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) \leq \eta < \epsilon.$$

Thus

$$A_1 \subset \{\gamma : \sup_{z \in D \cap B} \inf_{x \in B_d(z, \epsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) < \epsilon\}.$$

Since $A_1 \in \mathbb{F}(I)$ hence we have the required results. □

Definition 3.4. Let $\langle\langle D_\gamma, u_\gamma \rangle\rangle_{\gamma \in \Gamma}$ be a net of partial maps, $(D, u) \in \mathcal{P}[X, Y]$ and I be an ideal of Γ . We say that $\langle\langle D_\gamma, u_\gamma \rangle\rangle_{\gamma \in \Gamma}$ is I -converges pointwise to (D, u) if whenever $x \in D_\gamma$ for a cofinal subset $\Gamma_0 \subset \Gamma$, then $x \in D$ and $u(x) = I_{\Gamma_0} - \lim u_\gamma(x)$.

Proposition 3.5. Let $\langle\langle D_\gamma, u_\gamma \rangle\rangle_{\gamma \in \Gamma}$ be a net in $\mathcal{P}[X, Y]$. If the net is $\mathcal{P}_I^+(\mathcal{B})$ -convergent to $(D, u) \in \mathcal{C}[X, Y]$, then it is pointwise I -convergent to (D, u) .

Proof. Let Γ_0 be a cofinal set of Γ and let $\gamma \in \Gamma_0$ with $x \in D_\gamma$. By proposition 3.1, we have $x \in D$ because D is closed. Again by continuity of u , we can choose $\delta < \frac{\epsilon}{2}$ such that if $d(z, x) < \delta$ then $\mu(u(z), u(x)) < \frac{\epsilon}{2}$. Also by $(B_I^*)^+$ -convergence of graphs

$$B = \{\gamma \in \Gamma_0 : (\{x\} \times Y) \cap Gr(u_\gamma) \subset Gr^\delta(u)\} \in \mathbb{F}(I_{\Gamma_0}).$$

Choose $\gamma \in B$, then there exists $z_\gamma \in D$ with

$$(d \times \mu)((x, u_\gamma(x))(z_\gamma, u(z_\gamma))) < \delta.$$

So we have $\mu(u_\gamma(x), u(z_\gamma)) < \delta$ and $d(x, z_\gamma) < \delta$. Again by triangle inequality

$$\mu(u_\gamma(x), u(x)) \leq \mu(u_\gamma(x), u(z_\gamma)) + \mu(u(z_\gamma), u(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence $B \subset \{\gamma : \mu(u_\gamma(x), u(x)) < \epsilon\}$. Since $B \in \mathbb{F}(I_{\Gamma_0})$ thus the later set. This completes the proof. \square

Theorem 3.6. *Let \mathcal{B} be a bornology on (X, d) and let (D, u) be strongly uniformly continuous relative to \mathcal{B} with values in (Y, μ) . Then a net $\langle \langle D_\gamma, u_\gamma \rangle \rangle_{\gamma \in \Gamma}$ in $\mathcal{P}[X, Y]$ is $\mathcal{P}_I^+(\mathcal{B})$ -convergent to (D, u) is equivalent to the condition $\forall B \in \mathcal{B}$ and $\epsilon > 0$, there exists $\zeta > 0$ such that*

$$\left\{ \gamma : \sup_{z \in D_\gamma \cap B} \sup_{x \in B_d(z, \zeta) \cap D} \mu(u(x), u_\gamma(z)) < \epsilon \right\} \in \mathbb{F}(I).$$

Proof. Sufficient part of the statement follows from proposition 3.5. We only need to show that upper bornological I -convergence implies the sup-sup condition above.

Let $B \in \mathcal{B}$ and $\epsilon > 0$ be given. Let $\eta > 0$ be such that $2\eta < \epsilon$. By strong uniform continuity of u relative to \mathcal{B} , there exists δ , $0 < \delta < 2\eta$ such that $x, y \in D \cap B^\delta$ and $d(x, y) < \delta$ implies $\mu(u(x), u(y)) < \eta$. Again by assumption

$$A = \left\{ \gamma : D_\gamma \cap B \subset D^{\frac{\delta}{2}} \right\} \in \mathbb{F}(I)$$

and

$$B = \left\{ \gamma : \sup_{z \in D_\gamma \cap B} \inf_{x \in B_d(z, \frac{\delta}{2})} \mu(u(x), u_\gamma(z)) < \frac{\delta}{2} \right\} \in \mathbb{F}(I).$$

Thus $A \cap B \in \mathbb{F}(I)$. Choose $\gamma \in A \cap B$. Then for every $z \in B \cap D_\gamma$, there exists $x_z \in B_d(z, \frac{\delta}{2}) \cap D$ such that $\mu(u(x_z), u_\gamma(z)) < \frac{\delta}{2} < \eta$. But since $B_d(z, \frac{\delta}{2}) \subset B^\delta$, for every $x \in B_d(z, \frac{\delta}{2}) \cap D \subset B^\delta \cap D$ we have by strong uniform continuity $\mu(u(x), u(x_z)) < \eta$ because $d(x, x_z) < \delta$. Thus for every $x \in B_d(z, \frac{\delta}{2}) \cap D$

$$\mu(u(x), u_\gamma(z)) \leq \mu(u(x), u(x_z)) + \mu(u(x_z), u_\gamma(z)) < \eta + \eta = 2\eta$$

and hence

$$\sup_{z \in D_\gamma \cap B} \sup_{x \in B_d(z, \frac{\delta}{2}) \cap D} \mu(u(x), u_\gamma(z)) \leq 2\eta < \delta.$$

Choose $\zeta = \frac{\delta}{2}$ then we have

$$A \cap B \subset \left\{ \gamma : \sup_{z \in D_\gamma \cap B} \sup_{x \in B_d(z, \zeta) \cap D} \mu(u(x), u_\gamma(z)) < \epsilon \right\}.$$

Since $A \cap B \in \mathbb{F}(I)$ thus the later set. This yields to the prove. \square

Theorem 3.7. *Let \mathcal{B} be a bornology on (X, d) that is stable under small enlargement and let (D, u) be uniformly continuous relative to \mathcal{B} with values in (Y, μ) . Then a net $\langle \langle D_\gamma, u_\gamma \rangle \rangle_{\gamma \in \Gamma}$ in $\mathcal{P}[X, Y]$ is $\mathcal{P}_I(\mathcal{B})$ -convergent to (D, u) if and only if both the following two condition hold:*

(1) *for each $B \in \mathcal{B}$ and $\epsilon > 0$, there exists $\zeta > 0$ such that*

$$\left\{ \gamma : \sup_{z \in D_\gamma \cap B} \sup_{x \in B_d(z, \zeta) \cap D} \mu(u(x), u_\gamma(z)) < \epsilon \right\} \in \mathbb{F}(I) ;$$

(2) *for each $B \in \mathcal{B}$ and $\epsilon > 0$,*

$$\{\gamma : D \cap B \subset D_\gamma^\epsilon\} \in \mathbb{F}(I).$$

Proof. Necessity follows from that (D, u) uniformly is continuous relative to \mathcal{B} implies strong uniform continuity relative to \mathcal{B} , by proposition 3.1 and theorem 3.2. Let $B \in \mathcal{B}$ and $\epsilon > 0$ be given, we only need to show that

$$\left\{ \gamma : \sup_{z \in D \cap B} \inf_{x \in B_d(z, \epsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) < \epsilon \right\} \in \mathbb{F}(I).$$

Choose $\zeta < \epsilon$ so small that $B^\zeta \in \mathcal{B}$ and satisfies

$$C = \left\{ \gamma : \sup_{z \in D_\gamma \cap B^\zeta} \sup_{x \in B_d(z, \zeta) \cap D} \mu(u(x), u_\gamma(z)) < \frac{\epsilon}{2} \right\} \in \mathbb{F}(I)$$

and

$$E = \{\gamma : D \cap B \subset D_\gamma^\zeta\} \in \mathbb{F}(I).$$

Choose $\gamma \in C \cap E$ and let $z \in D \cap B$, so there exists $x(z, \gamma) \in D_\gamma \cap B^\zeta$ with $d(z, x(z, \gamma)) < \zeta$. Again $\gamma \in C$, $x(z, \gamma) \in D_\gamma \cap B^\zeta$ and $z \in B_d(z, \zeta) \cap D$ implies $\mu(u(z), u_\gamma(x(z, \gamma))) < \frac{\epsilon}{2}$. Hence $\inf_{x \in B_d(z, \zeta) \cap D_\gamma} \mu(u(z), u_\gamma(x(z, \gamma))) < \frac{\epsilon}{2}$. Also

$$\sup_{z \in D \cap B} \inf_{x \in B_d(z, \epsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) \leq \sup_{z \in D \cap B} \inf_{x \in B_d(z, \zeta) \cap D_\gamma} \mu(u(z), u_\gamma(x)) \leq \frac{\epsilon}{2} < \epsilon.$$

Therefore $C \cap E \subset \left\{ \gamma : \sup_{z \in D \cap B} \inf_{x \in B_d(z, \epsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) < \epsilon \right\}$. Now $C \cap E \in \mathbb{F}(I)$. So the required result. \square

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